Course Stabilty of Structures

Lecture notes 2015.03.06 about

- 3D beams, some preliminaries (1:st order theory)
- Torsion, 1:st order theory
- 3D beams 2:nd order theory
- Torsional buckling
- Coupled buckling modes, examples
- Numerical calculation method

/Per J. G.

Pelarknäckning,
i-planet knäckning,
böjknäckning.
In-plane column buckling.

Vridknäckning.
Torsional buckling.

Böjvridknäckning.
Bending-torsional buckling?

Vippning, kantring
Lateral-torsional buckling. Tilting?
3D beams, some preliminaries

Coordinate system (right hand side orientation):

Stresses in beam (as considered in governing equations):

\[ \sigma_y = \sigma_z = \tau_{yz} = 0 \]
\[ \sigma_x \neq 0, \tau_{xy} \neq 0 \text{ and } \tau_{xz} \neq 0 \] (in general)

Cross section reference points and directions:
1:st order governing equations for 3D Bernoulli/Euler/StVenant/Vlasov beam

\[
\begin{align*}
(EAu')' &= -q_x - N'_o & \text{bar-action} \\
(Elv'')' &= q_y - m'_z - M'_z & \text{bending in x-y plane} \\
(Elw''')' &= q_z + m'_y + M'_{yo} & \text{bending in x-z plane} \\
(El_{o\phi'}')' - (GK_{V\phi})' &= m_x & \text{torsion}
\end{align*}
\]

The equations are linear and uncoupled.

\(v = v(x_o, y_o)\) and \(w = (x_o, y_o)\). Thus \(v(x, y) = v(-z-z_o)\phi\) and \(w(x, y) = w+(y-y_o)\phi\).
**Torsion**

**Kinematics:**

1) In the $y$-$z$ plane, the beam cross-section moves as a rigid body rotating around the rotation center.

2) Displacement $u$ is not equal in different parts of a cross section: warping.
Two torsion deformation modes:
The magnitude of warping may be constant or vary along the beam.
These two cases corresponds to different stress and deformation performance.

a) (“St Venant theory”) The warping is constant along the beam. This is the case
if the torque is constant and there is nothing preventing warping at the ends
of the beam.

In this case \( \sigma_x = \varepsilon_x = 0 \). Twisting and warping is due to \( \gamma_{xy} \neq 0 \) and \( \gamma_{xz} \neq 0 \).

\[
\frac{\varphi(L) - \varphi(0)}{L} = \frac{d\varphi}{dx} = \varphi' = \frac{T}{GK_v}
\]

For distributed load, equilibrium for a part \( dx \) gives \( dT + m_x dx = 0 \), i.e. \( T' = -m_x \):

\[
(GK_v \varphi')' = -m_x \quad (5)
\]

To be strict, \( m_x \neq 0 \) should not be allowed since it gives varying \( T \) and warping, and
therefore contradicts the theory.

The case of constant warping is the “St Venants torsion theory”. Cross section
geometry parameter \( K_v \) and shear stress distribution are in this theory
determined by an Poisson’s equation, derived from definition of shear strain,
\( \tau = G\gamma \) and \( \sigma_x = 0 \).
b) (“Vlasov theory”) The warping is varying along the beam. This is the case if the torque is varying or warping is prevented at an end of the beam.

In “pure Vlasov torsion” is $\gamma_{xy}=0$ and $\gamma_{xz}=0$, and twisting is due to $\varepsilon_x \neq 0$.

Prevention of warping is of great importance the torsional stiffness of beams with certain thin-walled cross-sections, eg:

For most other cross-sections is the influence (on torsional stiffness) of prevented warping disregarded.

The pure Vlasov torsion gives:

\[
(EI \omega \phi'')'' = m_x
\] (6)
An illustration

Prevented warping ("Vlasov"). Twist of beam due to normal stress $\sigma_x$ and strain $\varepsilon_x$ giving bending deformation of upper and lower flanges.

Free warping ("St Venant"). Twist of beam due to shear stress and strain giving "spiral shape" of the web and each of the two flanges.

Superposition of the two stress-systems gives

"the governing equation for mixed torsion" acc. to 1:st order theory:

$$ (EI\omega \phi^\prime\prime\prime\prime - (GK_v \phi^\prime)^\prime) = m_x \tag{7} $$

Boundary conditions if $l_\omega = 0$
- $\phi = 0$ (or other known value)
- $T$ known, giving $\phi^\prime = T/(GK_v)$

Boundary conditions if $l_\omega \neq 0$
- $\phi = 0$ (or other known value)
- $\phi^\prime = 0$ (prevented warping, clamped)
- $\phi^\prime\prime = 0$ (free warping (or known normal stress giving some $\phi^\prime\prime \neq 0$))
- $T$ known, giving $-(EI\omega \phi^\prime\prime) + GK_v \phi^\prime = T$
2:nd order governing equations for 3D Bernoulli/Euler-/StVenant/Vlasov beam

**Assumptions**
- Linear elastic material
- Constant $EA$, $EI$, $GK_v$, $El_w$ and $N$ along beam
- No initial stress and no initial curvature (imperfections)
- No distributed moment loads
- $N$ known and no influence of shear on normal axial force

\[
\begin{align*}
EAu'' &= q_x \quad \text{bar-action} \quad (8) \\
El_z v'''' &= q_y + N(v'' + z_o \varphi '') - (M_y^0 \varphi '') \quad \text{bending in x-y plane} \quad (9) \\
El_v w'''' &= q_z + N(w'' - y_o \varphi '') - (M_z^0 \varphi '') \quad \text{bending in x-z plane} \quad (10) \\
El_w \varphi '''' - GK_v \varphi '' &= m_x + (2:nd \text{ order terms}) \quad \text{torsion} \quad (11)
\end{align*}
\]

where (2:nd order terms)=

\[
\begin{align*}
&= N(l_o/A) \varphi '' - N(y_o w'' - z_o v'') \ldots \\
&- (M_y^0 v')' + (V_z^0 v)' - (M_z^0 w')' - (V_y^0 w)' \ldots \\
&- 2(M_y^0 \varphi ')z_o + 2(M_z^0 \varphi ')y_o \ldots \\
&+ q_z v - q_y w - q_z (z_1 - z_o) \varphi - q_y (y_1 - y_o) \varphi 
\end{align*}
\]

Notations: see next page.

Notations

$(y_0, z_0)$ is the location of the rotation center

$(y_1, z_0)$ is the location of load $q_y$

$(y_0, z_1)$ is the location of load $q_z$

$I_0 = I_y + I_z + (y_0^2 + z_0^2)A$ is the polar moment of inertia with respect to the rotation center

$V_y^o$, $V_z^o$, $M_y^o$, $M_z^o$ are section shear forces and moments acc. to 1:st order theory
Outline of derivation of governing equations for 3D beam, 2:nd order

The three “beam stress” components $\sigma_x$, $\tau_{xy}$ and $\tau_{xz}$, acting on a small volume $dx dy dz$ in the beam are considered:

Equilibrium in $y$- and $z$-directions:

\[
\begin{align*}
\tau'_{xy} + U_y + (\sigma_x v')' &= 0 \\
\tau'_{xz} + U_z + (\sigma_x w')' &= 0
\end{align*}
\]  

(12a,b)

$U_y$ and $U_z$ are load/volume in the $y$- and $z$-directions.

2:nd order effects of the shear stresses are not considered, just as in the 2D beam analysis.

The 2:nd order equations can obtained from the 1:st order equation by adding the second order component of the normal stress to the loads in the $y$- and $z$-directions (loads/length $q_y$ and $q_z$):

- replacing load $q_y$ with load $q_y + (\int_{A} \sigma_x v' dA)'$
- replacing load $q_z$ with load $q_z + (\int_{A} \sigma_x w' dA)'$

(In case of torsion with prevented warping additional considerations are needed.)

Note: $\sigma_x$ is the normal stress in direction $(1, v', w')$, that is in the direction along a line in the beam which in the unloaded beam was oriented parallel with $x$-axis. This direction of $\sigma_x$ is thus not the $x$-direction and in general neither perpendicular to the surface on which $\sigma_x$ acts. The latter is of relevance for St Venant torsion.
**Equations (8-11) in the special case of double symmetric and massive (thickwalled) beam cross sections:**

Double symmetric: \( y_0 = z_0 = 0 \)

Massive (thick walled): \( y_0 \approx z_0 \approx 0 \)

\[
\begin{align*}
\text{EAu''} &= q_x \\
\text{El}_z v'''' &= q_y + Nv'' - (M_y^{(0)} \varphi)'' \\
\text{El}_y w'''' &= q_z + Nw'' - (M_z^{(0)} \varphi)'' \\
\text{El}_w \varphi'''' - GK_v \varphi'' &= m_x + (2:\text{nd order terms})
\end{align*}
\]

where (2:nd order terms) =

\[
= N(I_o/A) \varphi'' \quad \ldots
- (M_y^{(0)} v')' + (V_z^{(0)} v)' - (M_z^{(0)} w')' - (V_y^{(0)} w)' \ldots
+ q_z v - q_y w - q_z z_1 \varphi - q_y y_1 \varphi
\]

For the I-beam section is \( I_\omega \neq 0 \).

For all other is \( I_\omega \approx 0 \).
If moreover no distributed load \((q_x=q_y=q_z=m_x=0)\):

\[
\begin{align*}
&E Au'' = 0 \\
&El_z v''' = Nv'' - (M_y^0 \phi)'' \\
&El_y w''' = Nw'' - (M_z^0 \phi)'' \\
&El_\omega \phi''' = G K v \phi'' = N (I/O/A) \phi'' - (M_y^0 v')' + (V_z^0 v)' - (M_z^0 w')' - (V_y^0 w)' \\
&\text{.....(14a,b,c,d)}
\end{align*}
\]

And for stability analysis with respect only to compressive force \(-N\):

\[
\begin{align*}
&El_z v''' = -Nv'' = 0 \\
&El_y w''' = -Nw'' = 0 \\
&El_\omega \phi''' = (G K v + N I/O/A) \phi'' = 0
\end{align*}
\]

Eq \((15a,b,c)\) have the nice feature of being uncoupled.
**Torsional buckling**

Only double symmetric and/or thickwalled cross sections considered.

**Governing equation and solution for beams with \( I_\omega \neq 0 \):**

\[
E I_\omega \varphi''' - (G K_v + N I_o / A) \varphi'' = 0
\]

\[
\varphi = C_1 \cos(\lambda x) + C_2 \sin(\lambda x) + C_3 x + C_4
\]

for \((G K_v + N I_o / A) < 0\) (compression), \(\lambda = \sqrt{- (G K_v + N I_o / A) / E I_\omega}\)

\[
\varphi = C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x) + C_3 x + C_4
\]

for \((G K_v + N I_o / A) > 0\) (tension or small compression), \(\lambda = \sqrt{(G K_v + N I_o / A) / E I_\omega}\)

\[
(15c)
\]

One solution for homogeneous boundary conditions is \(C=0\).

From the boundary conditions: \(A C=0\).

By investigation of \(\text{det}(A)=0\) can possible \(N_{cr}\) for the actual boundary conditions be determined.

**Governing equation and solution for beams with \( I_\omega = 0 \):**

\[
(G K_v + N I_o / A) \varphi'' = 0
\]

\[
v = C_1 x + C_2
\]

One solution for homogeneous boundary conditions (\(\varphi=0\) and/or \(\varphi'=0\)) is \(C=0\).

Other solutions are possible only if

\[
(G K_v + N I_o / A) = 0
\]

\[
(19)
\]

Thus the critical load for torsional buckling is

\[
N_{cr} = - G K_v A / I_o
\]

\[
(20)
\]

The curve \(\varphi(x)\) may have any continuous shape in the case torsional buckling of a beam with \(I_\omega=0\)! (Provided that the boundary values at \(x=0\) and \(X=L\) are fulfilled.)
Further studies of torsional buckling:
Thin walled pipe

Cross section area = $A$
Radius = $R$
Wall thickness = $t<<R$

Potential energy, $\pi$

\[
\begin{align*}
\pi &= -\sigma ah(1-\cos\gamma) + (1/2)\tau\gamma V \quad \text{where} \quad a h = V \quad \text{and} \quad \tau = G\gamma \\
\pi' &= -\sigma \sin\gamma + G\gamma \\
\pi'' &= -\sigma \cos\gamma + G
\end{align*}
\]

For $\gamma=0$ is $\pi'=0$ (equilibrium) and $\pi''=0$ (neutral equilibrium) found for $\sigma_{cr} = G$ (a simple and useful relation!?) \hspace{1cm} (21)

For the pipe this means

$N_{cr} = -GA$ \hspace{1cm} (22)

Compare with Eq (20) for a thin walled pipe:

$N_{cr} = -GK_v A/I_o = -GI_o A/I_o = -GA$
The result $\sigma_{cr} = G$ should be possible to derive also by means of equilibrium analysis. How?
Further studies of torsional buckling:
Derivation of buckling load at St Venant torsion (beams with $I_\omega=0$)

Potential energy, $\pi$

$$\pi = -\int_{A} \sigma dAL(1-\cos \gamma) + \frac{T\varphi}{2}$$

where $\gamma = r\varphi/L$, $T = GK_v\varphi/L$ and $\cos \gamma \approx 1 - \gamma^2/2$

$$\left\{ \begin{array}{l}
\pi = -\sigma L \int_{A} \frac{(r\varphi/L)^2}{2} dA + \frac{GK_v\varphi^2}{2L} = -\frac{\sigma \varphi^2}{2L} \int_{A} r^2 dA + \frac{GK_v\varphi^2}{2L} \\
\pi' = -\frac{\sigma \varphi}{L} I_o + \frac{GK_v\varphi}{L} \\
\pi'' = -\frac{\sigma}{L} I_o + \frac{GK_v}{L}
\end{array} \right.$$

For $\varphi=0$ is $\pi'=0$ (equilibrium) and $\pi''=0$ (neutral equilibrium) found for

$$\sigma_{cr} = \frac{GK_v}{I_o} \text{ giving}$$

$$N_{cr} = -GK_vA/I_o \quad \text{which is Eq (20)}$$

The similar energy based derivation of critical load for the case $I_\omega\neq0$ can be made, but then variation of $\varphi$ along the beam must be considered (by means of Eq (15a)).
Lateral torsional buckling. Example 1

\[ M_y(x) = \text{constant} = M \]

No other loads
\[ y_0 = z_0 = 0 \]

Eq. (9), (10) and (11).

\[
\begin{align*}
&EI_z \dddot{v} + M \ddot{\varphi} = 0 & a) \\
&EI_y \dddot{w} = 0 & b) \text{ not homogenous} \\
&EI_0 \dddot{\varphi} - GJ_0 \ddot{\varphi} + M \ddot{v} = 0 & c)
\end{align*}
\]

B.c. : \( v(0) = v(L) = v''(L) = 0 \)
\( \varphi(0) = \varphi''(0) = \varphi(L) = \varphi''(L) = 0 \)

Assume : \( v = C_1 \sin \frac{\pi x}{L} \), \( \varphi = C_2 \sin \frac{\pi x}{L} \)

Check b.c. : OK!

Check eq. 3.

\[
\begin{align*}
&EI_z C_1 \left( \frac{\pi}{L} \right)^4 \sin \left( \frac{\pi x}{L} \right) - MC_2 \left( \frac{\pi}{L} \right)^2 \sin \left( \frac{\pi x}{L} \right) = 0 \\
&EI_0 C_2 \left( \frac{\pi}{L} \right)^2 \sin \left( \frac{\pi x}{L} \right) + GJ_0 C_2 \sin \left( \frac{\pi x}{L} \right) = 0 \\
& - MG \left( \frac{\pi}{L} \right)^2 \varphi \left( \frac{\pi x}{L} \right) = 0
\end{align*}
\]

OK! (Valid for all x)
\[
\begin{bmatrix}
\frac{EI_z}{L^2} & 0 \\
0 & -M
\end{bmatrix}
\begin{bmatrix}
\frac{EI_0}{L^2} - G\kappa
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \end{bmatrix}
\]

\[C_1 = C_2 = 0 \text{ or } \det \left[ \frac{EI_0}{L^2} - G\kappa \right] = 0 \Rightarrow \]

\[EI_z \frac{D^2}{L^2} \left( EI_0 \frac{D^2}{L^2} + G\kappa \right) - M^2 = 0 \]

\[M = M_{cr} = \frac{EI_z}{L} \sqrt{EI_0 \left( EI_0 \frac{D^2}{L^2} + G\kappa \right)} \]
Lateral torsional buckling. Example 2.

\[ V_y(x) = -F \]
\[ M_z^0(x) = F(-x + L) \]
\[ y_0 = z_0 = 0 \]
\[ (y, z_1) = 0 \]

\[ (9), (10) \text{ and } (11): \]

\[ \begin{cases} \left( EI_z \psi'''' + F((-x + L) \theta)'' = 0 \right) \quad (a) \\ -(K_\theta \theta'' + F((-x + L) \psi)' - Fw' = 0 \quad (b) \end{cases} \]

- (a) and (b) coupled \( \Rightarrow \) torsion + bending buckling.
- One solution is \( \nu = \theta = 0 \), other sol. \( \Rightarrow \) buckling
- (a) and (b) have varying coefficients \( \Rightarrow \)
  - numerical solution required

(a) and (b) can be rewritten as one equation:

\[ \theta'' + \frac{F^2}{\xi K_\theta E I_y} (L-x)^2 \theta = 0 \quad \theta(0) = 0, \quad \theta(L) = 0 \]
Lateral torsional buckling, Ex. 3

$\theta_1 \neq 0, \theta_2 \neq 0, m_x \neq 0$

Eq. (9), (10) and (11) →

No buckling!

Unique deflection and rotation already at small load.
Numerical solution of diff. eq. by point collocation method, applied to stability analysis.

Example

Diff. eq.:

\[(EI(x)u''(x))'' - (N(x)u'(x))' = 0\]

Assume:

\[u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \ldots + \alpha_7 x^6\]

4 b.c. + 3 collocation points give

\[\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_7\]

The points can be

\[x = 1/4, \; x = 1/2 \text{ and } x = 3L/4\]

continued next page....
\[
\begin{cases}
V(0) = 0 \\
V''(0) = 0 \\
V(L) = 0 \\
V''(L) = 0 \\
(EI(\frac{L}{4}) V^{''} (\frac{L}{4}))^{\prime} - (N (\frac{L}{4}) V^{'} (\frac{L}{4}))^{\prime} = 0 \\
(EI(\frac{L}{2}) V^{''} (\frac{L}{2}))^{\prime} - (N (\frac{L}{2}) V^{'} (\frac{L}{2}))^{\prime} = 0 \\
(EI(\frac{3L}{4}) V^{''} (\frac{3L}{4}))^{\prime} - (N (\frac{3L}{4}) V^{'} (\frac{3L}{4}))^{\prime} = 0 \\
\end{cases}
\]

\[
det \left[ \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\end{array} \right] = 0 \Rightarrow \nu_{cr}
\]