Stability of structures
FE-based stability analysis
Non-linear geometry, example

\[ \frac{P}{EA} \left( \frac{l_0}{a} \right)^3 \]

\( P = 0 \)

\( -P \)
Non-Linear geometry, example
- kinematics

The lengths of the bar in undeformed and deformed configurations: (Truncated Taylor expansion)

\[ l_0 = \sqrt{b^2 + a^2} \approx b \left( 1 + \frac{1}{2} \frac{a^2}{b^2} \right) \]

\[ l_1 = \sqrt{b^2 + (a + u)^2} \approx b \left( 1 + \frac{1}{2} \left( \frac{a + u}{b} \right)^2 \right) \]

By insertion of the lengths, the strains may be written as:

\[ \varepsilon = \frac{l_1 - l_0}{l_0} \approx \frac{a}{l_0 l_0} + \frac{1}{2} \left( \frac{u}{l_0} \right)^2 \]
Non-Linear geometry, example - equilibrium

Choosing a linear elastic material: \[ N = A \sigma = EA \varepsilon \]

\[ N = EA \varepsilon \approx EA \left( \frac{a}{l_0} \frac{u}{l_0} + \frac{1}{2} \left( \frac{u}{l_0} \right)^2 \right) \]

Equilibrium of the central node:

\[ P = 2N \frac{a + u}{l_1} \approx \frac{2EA}{l_0^3} \left( au + \frac{1}{2} u^2 \right) (a + u) \]

since \( \sin \theta = (a+u)/L_1 \)

and \( l_1 \approx b \left( 1 + \frac{1}{2} \left( \frac{a + u}{b} \right)^2 \right) \)
Non-Linear geometry, example

\[ P = 2N \frac{a + u}{l_1} \]

Tangential stiffness: \[ K_t = \frac{dP}{du} \]

Derivation of the equilibrium equation:

\[ K_t = \frac{d}{du} \left( 2N \frac{a + u}{l_0} \right) = 2\frac{EA}{l_0} \left( \frac{a + u}{l_0} \right)^2 + 2\frac{N}{l_0} \]

Final form of tangential stiffness:

\[ K_t = 2\frac{EA}{l_0} \left( \frac{a}{l_0} \right)^2 + 2\frac{EA}{l_0} \frac{2au + u^2}{l_0^2} + 2\frac{N}{l_0} \]

\[ = K_0 + K_u + K_\sigma \]

\[ K_u = K_u(u) \]

\[ K_\sigma = K_\sigma(\sigma) \]
Non-Linear geometry, example

- First order theory: \( K_t = K_0 \)
- Second order theory: \( K_t = K_0 + K_\sigma \)
- Third order theory: \( K_t = K_0 + K_\sigma + K_u \)

\[
K_t = 2 \frac{EA}{l_0} \left( \frac{a}{l_0} \right)^2 + 2 \frac{EA}{l_0} \frac{2au + u^2}{l_0^2} + 2 \frac{N}{l_0} \\
= K_0 + K_u + K_\sigma
\]

\[
K_u = K_u(u) \\
K_\sigma = K_\sigma(\sigma)
\]
General bar element
see: S. Krenk, Non-Linear Modeling and Analysis of Solids and Structures

\[
K_0 = \frac{EA}{L} \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

First order:
\[K_t = K_0\]
bar2e.m in Calfem

\[
K_\sigma = \frac{N}{L} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\]

Second order:
\[K_t = K_0 + K_\sigma\]
bar2g.m in Calfem

\[
K_\sigma = \frac{EA}{L^3} \begin{bmatrix}
b_u & -b_u \\
-b_u & b_u
\end{bmatrix}
\]

Third order:
\[K_t = K_0 + K_\sigma + K_u\]
Not in Calfem

where
\[
b_u = \begin{bmatrix}
\Delta u_x (2a + \Delta u_x) & a\Delta u_y + b\Delta u_x + \Delta u_x \Delta u_y \\
\Delta u_y (2a + \Delta u_y) & a\Delta u_x + b\Delta u_y + \Delta u_y \Delta u_x
\end{bmatrix}
\]

and
\[
\begin{align*}
a &= (x_2 - x_1) \\
b &= (y_2 - y_1) \\
\Delta u_x &= (u_3 - u_1) \\
\Delta u_y &= (u_4 - u_2)
\end{align*}
\]
General solid element
see: S. Krenk, Non-Linear Modeling and Analysis of Solids and Structures

The tangential element stiffness for solid elements may in many cases also be written on the form:

- First order theory: \( K_t = K_0 \)
- Second order theory: \( K_t = K_0 + K_\sigma \)
- Third order theory: \( K_t = K_0 + K_\sigma + K_u \)
Stability - Linear Buckling - example

Bar with equilibrium in deformed configuration only:

Second order theory: \( K_t = K_0 + K_\sigma \)

\[
K_T = \frac{EA}{L} \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} + \frac{N}{L} \begin{bmatrix}
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 \\
\end{bmatrix}
\]

Note! \( N = -P \) and the second term becomes negative:

\[
K_T = \begin{bmatrix}
\frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\
0 & -\frac{P}{L} & 0 & \frac{P}{L} \\
-\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\
0 & \frac{P}{L} & 0 & -\frac{P}{L} + k_f \\
\end{bmatrix}
\]

\( u_1 = u_2 = 0 \)

\[
K_T = \begin{bmatrix}
\frac{EA}{L} & 0 \\
0 & -\frac{P}{L} + k_f \\
\end{bmatrix}
\]

Tangent stiffness \( K_t = 0 \) when \( \det(K_t) = 0 \)

\[
\det(K_t) = 0 \quad \Rightarrow \quad P = k_f L
\]
Stability - Linear Buckling
- Classical problem

• Look for displacements \( a \) when the tangent stiffness becomes zero:

\[
K_C \ a = 0
\]

where \( K_C = K_0 + K_\sigma \) is the tangent stiffness in the current state. This is a homogeneous equation system with non-trivial solutions \( a \).

• In classical buckling analysis the current state is the unloaded base state.

• A homogeneous equation system may be formulated as an eigenvalue problem:

\[
(K_0 + \lambda_i K_\sigma )x_i = 0
\]

\( \lambda_i = \text{the eigenvalues (force multipliers)} \)

\( x_i = \text{the buckling mode shapes} \)

• If the current state is the unloaded state, solve the second-order system for loads \( f \) to get the stress distribution in the structure.

• The critical load \( f_{cr} = \lambda_i f \) (\( \lambda_i \) becomes the load multiplier)
Example: classical buckling

- Simple frame
- Unloaded base state
- Differential load = -1N in y-dir at top of both pillars
- Fixed supports at base
Example: classical buckling

1st eigenvalue = 2.00 \times 10^6
Critical load \( f_{cr} = 2.00 \times 10^6 \times (-1) \) N
\( f_{cr} = -2.00 \times 10^6 \) N

2nd eigenvalue = 3.73 \times 10^6
Critical load \( f_{cr} = 3.73 \times 10^6 \times (-1) \) N
\( f_{cr} = -3.73 \times 10^6 \) N
Classical Linear Buckling in ABAQUS

- Apply loads, (for example 1 N) and boundary conditions
- Choose ”Linear Perturbation” and then ”Buckle” as the step.
  (Give number of eigenvalues that you want, the first (lowest) eigenvalue gives the first buckling mode)
- Apply boundary conditions.
- Solve the eigenvalue problem.
- The solution gives the buckling modes and the force multipliers $\lambda_i$ for the buckling loads.
- $f_{cr} = \lambda_i f$ will then give the buckling loads.
Stability - Linear Buckling
- General problem

- If the current state is caused by pre-loads, $f_{pre}$

$$K_C = K_0 + K_\sigma + K_u$$

is the tangent stiffness caused by the pre-loads.

- A homogeneous equation system may still be found as an eigenvalue problem:

$$(K_c + \lambda_i K_\sigma)x_i = 0$$

$\lambda_i =$ the eigenvalues (force multipliers)

$x_i =$ the buckling mode shapes

- $K_\sigma$ is now the differential stiffness at this state caused by the loads $\Delta f$.

- The critical load is now $f_{cr} = f_{pre} + \lambda_i \Delta f$ (where $\lambda_i$ is the load multiplier solved by the eigenvalue problem)

- If geometric nonlinearity is included, the base state geometry is the deformed geometry at the end of the last step.
Example: general linear buckling

- Same frame
- Preload with $-1.9 \times 10^6$ N in y-dir at top of both pillars
- Differential load = -1N in y-dir at top of both pillars
- Fixed supports at base
Example: general linear buckling

1st eigenvalue = 0.11 \times 10^6
Critical load \( f_{cr} = (-1.9 + 0.11 \times (-1)) \) \times 10^6 N
\( f_{cr} = -2.01 \times 10^6 \) N

2nd eigenvalue = 1.84 \times 10^6
Critical load \( f_{cr} = (-1.9 + 1.84 \times (-1)) \) \times 10^6 N
\( f_{cr} = -3.74 \times 10^6 \) N
Example 2: general linear buckling

- Same frame
- Preload with $-1.9 \times 10^6$ N in y-dir at top of both pillars and a load at the left top corner of 30 kN in negative x-dir
- Differential load = -1N in y-dir at top of both pillars
- Fixed support at base
Example 2: general linear buckling

Pre-load $-1.9 \times 10^6$ N in y-dir and $-20$ kN in x-dir

1st eigenvalue = 0.15 $10^6$

$f_{cr} = -2.05 \times 10^6$ N (+ stresses from pre-load)

Pre-load $-1.9 \times 10^6$ N in y-dir and $-40$ kN in x-dir

1st eigenvalue = 0.26 $10^6$

$f_{cr} = -2.16 \times 10^6$ N (+ stresses from pre-load)
General Linear Buckling in ABAQUS

• First, create a general static step (and non-linear geometry if desired)
• Create ”Linear Perturbation” and then ”Buckle” as the second step.
• Apply pre-loads and boundary conditions in the first step (general static step)
• Apply loads and boundary conditions in the second step (buckle), (for example 1 N)
• Solves first the pre-load step and then the eigenvalue problem with the base state from the pre-load.
• \[ f_{cr} = f_{pre} + \lambda_i \Delta f \] give the buckling loads.
Stability - Non-linear Buckling

• Element stiffness calculated with equilibrium in deformed configuration and updated displacement stiffness:

Third order theory: \( K_t = K_0 + K_\sigma + K_u \)

• Includes all static effects in a physical problem.
• Loading may be made until collapse is reached and post-buckling analysis may be performed.
Solution of Non-linear Equations

Direct explicit method:

\[ \Delta u_n = K_t^{-1} \Delta P_n \]

\[ u_{n+1} = u_n + \Delta u_{n+1} \]

\( R \): residual, additive error

Divide into a number of load-steps
Out-of Balance Forces

- External forces: $P$
  \[ P = f_b + f_l \]
- Internal forces: element forces = $I$
  \[ I = \int_A tB^T DB \, dA \quad a = \int_A tB^T \sigma \, dA \]
- Equilibrium: $P-I=0$
- In the direct explicit method: $P-I=R$
- $R$: Force Residual (Out-of-balance forces)
Newton-Raphson Method

Load steps $n=1, 2, ...$

\[ P_n = P_{n-1} + \Delta P_n \]
\[ u_n^0 = u_{n-1} \]

Iterations $i=1, 2, ...$

calculate \( \varepsilon_n \) from \( u_n^i \)

calculate residual \( R_n^i = P_{n-1}^i - I_n^i \)

calculate \( K_{tn}^{i-1} \)

\[ \delta u_n^i = (K_{tn}^{i-1})^{-1} R_n^i \]
\[ u_n^i = u_n^{i-1} + \delta u_n^i \]

stop iteration when residual is ok

end of load
Example 1: Non-linear buckling

- Same frame
- Load with $-2.5 \times 10^6$ N in y-dir at top of both pillars
- Fixed support at base
- Solve with non-linear geometry

- No buckling!!!
- Why?
Example 2: Non-linear buckling, imperfections

Load \(-2.5 \times 10^6\) N in y-dir and \(-1\) kN in x-dir (top left corner)

No buckling!!

Load \(-2.5 \times 10^6\) N in y-dir and \(-10\) kN in x-dir (top left corner)

Buckling at \(t=0.926\)

\[ f_{cr} = -2.35 \times 10^6 \text{ N} \]
Stability with imperfections

• General types of imperfections may be added to non-linear buckling analysis (2\textsuperscript{nd} or 3\textsuperscript{rd} order analysis)
  1. Through adding eigenmode shapes on structure
  2. Through adding deformation from a previous static analysis
Example 3: Non-linear buckling - displacement control, imperfections

Displacement -0.2m in y-dir and -1 kN in x-dir (top left corner)

Force at 0.2m displ = 2.09 $10^6$ N

force = 2.05 $10^6$ N
Non-linear Buckling in ABAQUS

- Apply a load larger than the anticipated buckling load
- Choose "Static, General" problem as the step.
- Choose Nlgeom: on
- The time is fictive, dividing the load into load increments.
- Apply boundary conditions.
- A solution may not be found when a buckling load is reached.
- Preferably use displacement control.
Examples
Snowloads on slender constructions
- and the finite element method
Ishall
- underspänd tre-ledstakstol

- Total last under en snörik vinter ca: 1.5 kN/m²
- Ska klara ca 3.4 kN/m²
- Spännvidd 48m
Unstable
Critical load $\sim 1.2 \text{ kN/m}^2$
Unstable
Critical load \( \sim 1.7 \text{ kN/m}^2 \)
Stable
Critical load $\sim 2.5 \text{ kN/m}^2$

But too high stresses
Another similar ice-arena (span 52m)
Measured imperfections
Buckling shape analysed with imperfections
Buckling due to heat from fire

Brandförlopp temp.~ tid