Dynamics of Structures
Elements of structural dynamics

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Overview

1. SDOF system
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   - MDOF SYSTEM
   - Solution of Equation of motion

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Some questions...

- How do structures behave during earthquakes?
- How can the seismic loads be represented and calculated?
- Can we assume structures behave in elastic range? Is it possible?
In order to study the dynamic behavior of a structure the simplest oscillating model is considered: **Singular degree of freedom – SDOF system**

- **M**: mass
- **k**: linear elastic lateral stiffness
- **c**: viscous damping
Equation of motion

The system consists of a mass $M$ on a spring (two columns) that remains in elastic range, $V = k \cdot u$, when it oscillates under a seismic acceleration $\ddot{x}_0(t)$. Defining $u$ as the relative displacement, the absolute acceleration $\ddot{u}$ is given by:

$$\ddot{x}(t) = \ddot{x}_0(t) + \ddot{u}(t)$$

Application of d’Alambert’s principle of dynamic equilibrium results in the equation of motion:

$$M\ddot{x} + c(\dot{x} - \dot{x}_0) + k(\dot{x} - \dot{x}_0) = 0 \quad \Rightarrow \quad M\ddot{u} + c\dot{u} + ku = -M\dddot{x}_0$$
Equation of motion

\[ M\ddot{u} + c\dot{u} + ku = -M\ddot{x}_0 \]

The equation can be rewritten dividing each term by \( M \):

\[ \ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = -\ddot{x}_0 \]

where:

\[
\begin{cases}
\omega_n = \sqrt{\frac{k}{M}} \quad \text{Natural circular frequency [rad/s]} \\
\xi = \frac{c}{2M\omega} \quad \text{Damping ratio [%]} 
\end{cases}
\]

Homogeneous equation: free vibration

\[ \downarrow \]

It can be neglected if \( \xi \ll 1 \)

Non Homogeneous equation: forced vibrations

\[
\begin{cases}
f_n = \frac{\omega_n}{2\pi} \quad \text{Natural frequency [Hz]} \\
T_n = \frac{1}{f_n} \quad \text{Natural period [s]} 
\end{cases}
\]
Solution of equation of motion

\[ \ddot{u} + 2\xi \omega_n \dot{u} + \omega_n^2 u = -\ddot{x}_0 \]

To solve the equation of motion the **Linear Time Invariant (LTI)** Systems theory can be used. A structure is a filter which transforms the input signal (ground acceleration) into an output signal (relative displacement, absolute acceleration, ecc.):

\[ \text{INPUT} \quad \ddot{x}_0 \quad \text{STRUCTURE} \quad T; \xi \quad \text{OUTPUT} \quad u; \dot{x} \]

The output signal can be calculated in frequency domain as the simple product of input signal and **frequency response function (FRF)** of the structure, or in the time domain as the convolution the input signal and the **impulse response function (IRF)** of the structure.
Solution of equation of motion in the time domain

In the time domain the convolution between the input signal and IIR of a structure is called **Duhamel’s integral**.

\[ u(t) = \frac{1}{\omega_d} \int_0^t \ddot{x}_0(\tau) e^{-\xi \omega_n (t-\tau)} \sin(\omega_d (t - \tau)) d\tau \]

where \( \omega_d = \omega_n \sqrt{1 - \xi^2} \)

After calculating the relative displacement, the relative velocity and the absolute acceleration can be obtained.

\[ \dot{u}(t) = \frac{du(t)}{dt} \]
\[ \ddot{x}(t) = -\omega_n^2 u(t) - 2\omega_n \xi \dot{u}(t) \]
Internal force and pseudo acceleration

Once we know the relative displacement in the time domain $u(t)$ it’s easy to calculate the internal force $F(t)$ as:

$$F(t) = ku(t) = \omega_d^2 u(t) = ma(t)$$

$a(t)$ is the pseudo acceleration

The pseudo acceleration is not the absolute acceleration!!!

$$\ddot{x} = -a(t) - 2\omega_n \xi \dot{u}$$

The seismic effects on the structure are calculated in each instant as a static force $F(t)$. In order to design or to assess a structure, we just need the peak force value.

$$F_{max} = \max|F(t)| = k \cdot \max|u(t)| = m \cdot \max|a(t)|$$
Internal force and pseudo acceleration

\[ F_{\text{max}} = \max |F(t)| = k \cdot \max |u(t)| = m \cdot \max |a(t)| \]

We just need the maximum value of displacement or pseudo-acceleration.

\[ \Downarrow \]

They depend on the dynamic properties of system \((T, \xi)\).

\[ \Downarrow \]

Given a ground acceleration, changing the natural period of the system and solving each time the Duhamel’s integral, we can calculated the maximum relative displacement or maximum pseudo-acceleration.

**Response Spectrum** \(S(T, \xi)\)
How can the response spectrum be calculated?

\[
T_1 = 0,25 \text{s} \quad T_2 = 0,5 \text{s} \quad T_3 = 1,0 \text{s}
\]

\[
\begin{aligned}
&u_{1,max} = S_{De}(T_1) \quad \text{Displacement Response Spectrum} \\
&a_{1,max} = S_{Ae}(T_1) \quad \text{Pseudo Acceleration Response Spectrum}
\end{aligned}
\]

\[
\begin{aligned}
&u_{2,max} = S_{De}(T_2) \\
&a_{2,max} = S_{Ae}(T_2)
\end{aligned}
\]

\[
\begin{aligned}
&u_{3,max} = S_{De}(T_3) \\
&a_{3,max} = S_{Ae}(T_3)
\end{aligned}
\]
Pseudo-Acceleration Response Spectrum $S_{A,e}(T)$

- For Rigid structures ($T = 0$) $S_a$ is equal to the maximum ground acceleration.
- For Flexible structures ($T \to \infty$) $S_a$ tends to zero.
- If the structure is characterized by a natural frequency similar to the ground motion one, the pseudo-acceleration is higher than PGA. The input signal is amplified!!!
An example

Let’s suppose a structure which can be modelled as a 1-DOF elastic and linear system with mass $M$ equal to 4000 tons (400 kN) and $K=630$ kN/mm. How can we calculate the maximum inertia force acting on the system if El Centro Earthquake occurs?

1. Calculating the natural period

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{4000 \cdot 10^3 \text{ kg}}{630 \cdot 10^6 \text{ N/m}}} = 0.5 \text{ s}$$

2. Deriving the maximum pseudo acceleration for the elastic spectrum $S(T)$: $1.05g$

3. Calculating the maximum force as:

$$F_{max} = m \cdot S_{A,e}(T) = 400 \cdot 1.05 = 420 \text{ kN}$$
Which ground motion do we select for the design of a structure?

We should calculate the maximum force for a lot of ground motions, using their response spectrum. For this reason European and National Standards give elastic response spectrum for each site depending on the ground type. It has been calculated by means of a probabilistic analysis considering a lot of real ground motions.

The standard response spectrum can be considered as the envelope of a lot of ground motion.
MDOF Systems

For most of structures the SDOF system can be considered as not accurate. We need to model the structure as a Multi degree of freedom M-DOF.

The number of degree of freedom for most of buildings is equal to 3 x number of storeys. In fact we can assume the masses concentrated in each floor which, if it is can be considered as infinitely rigid, is characterized by 3 degrees of freedom: 2 translation degrees of freedom (x,y) and 1 rotational degree of freedom.

The equation of motion can be written in matrix form:

\[
[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = -[M]\{1\}\ddot{x}_0(t)
\]

\([M] = \text{mass matrix}; [C] = \text{damping matrix}; [K] = \text{stiffness matrix}\)

\{\ddot{u}(t)\}; \{\dot{u}(t)\}; \{u(t)\}; \{1\} = \text{relative acceleration, velocity, displacement and influence nx1 vectors}\)
Solution of Equation of motion

\[
[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = -[M]\{1\}\ddot{x}_0(t)
\]

The equation represents a coupled system of second-order differential equation in which the independent variable is the time \( t \) and the dependent variables are the relative displacements. By coupled, it is meant that two or more of the dependent variables appear in each of the equations of the system. The most convenient method to solve the system is the so called modal superposition method.

The advantage is taken of the orthogonality properties of the mode shapes of the structure. The original equation of motion may be uncoupled by expressing the displacement vector as a linear combination of the structural mode shapes.

\[
\{u(t)\} = [\Phi]\{\eta(t)\}
\]

\([M] = \text{modal matrix}; \ [C] = \text{generalized coordinates vector}\)

**What is a mode shape?**
What is a mode shape?

Let consider an unforced and undamped structure. The equation becomes:

\[ [M]\{\ddot{u}(t)\} + [K]\{u(t)\} = 0 \]

The displacement vector (dependent variable of the differential equation) can be expressed as a linear combination of n solution, given by:

\[ \{u(t)\} = \{\Phi\} e^{i\omega t} \]

Substituting in the equation of motion we obtain:

\[ ( [K] - \omega^2 [M] ) \{\Phi\} = 0 \]

which admits a solution if the determinant of the matrix is equal to zero. This means to solve the following n order linear equation for the variable \( \omega^2 \).

\[ \left| [K] - \omega^2 [M] \right| = 0 \]

The solution gives us the \textbf{n circular natural frequency} \( \omega^2 \) of the structure which, substituting in the previous equation, give \( n \) vectors \( \{\Phi\} \), called \textbf{mode shapes}.
What is a mode shape?

A n-DOF system is characterized by n natural frequency (or period) and by n mode shapes. The first mode shape is associated with the highest natural period.

The free vibration of the structure can be seen a linear combination of all mode shapes.
Solution of Equation of motion

\[
[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = -[M]\{1\}\ddot{x}_0(t)
\]

Expressing the displacement vector as a linear combination of the structural mode shapes and substituting in the equation of motion we obtain:

\[
\begin{bmatrix}
\{u(t)\}
\end{bmatrix} = [\Phi]\{\eta(t)\}
\]

\[
[\Phi]^T[M][\Phi]\{\ddot{\eta}(t)\} + [\Phi]^T[C][\Phi]\{\dot{\eta}(t)\} + [\Phi]^T[K][\Phi]\{\eta(t)\} = -[\Phi]^T[M]\{1\}\ddot{x}_0(t)
\]

Thanks to the properties of the mass and stiffness matrix, we can decouple the equations into \( n \) uncoupled equation of motion.

\[
\ddot{\eta}(t) + 2\xi_i^* \omega_i \eta(t) + \omega_i^2 = -\frac{[\Phi]^T[M]\{1\}\ddot{x}_0(t)}{m_i^*} = -\gamma_i \ddot{x}_0(t) \quad \text{for } i = 1 \ldots n
\]

\( \omega_i \) = natural circular frequency for each \( i \) mode; \( \gamma_i \) = participation factor for each \( i \) mode;

**We have obtained \( n \) independent SDOF system!!!
Solution of Equation of motion

Each independent equation of motion of the i-th SDOF system can be solved by means of the Duhamel’s integral as seen previously.

\[ \ddot{\eta}(t) + 2\xi_i^* \omega_i \eta_i(t) + \omega_i^2 \eta_i(t) = -\frac{[\Phi]^T [M] \{1\} \ddot{x}_0(t)}{m_i^*} = -\gamma_i \ddot{x}_0(t) \quad \text{for } i = 1 \ldots n \quad \Rightarrow \eta_i(t) \]

We can evaluate maximum displacement and acceleration and force for each equation by means of RESPONSE SPECTRUM. But how can we get the maximum displacement, acceleration and force of the structure since the solution is a linear combination of the n independent solution?

\[ \{u(t)\} = \sum_i \{\Phi\}_i^T \eta_i(t) \]

The maximum displacement of the structure is in general less than the sum of maximum displacement associated to each mode i. In fact modes have different frequency of vibration and maximum displacement are not reached at the same time.

\[ \max \{u(t)\} \neq \sum_i \{\Phi\}_i^T \max(\eta_i(t)) \]
Solution of Equation of motion

In other words, one cannot consider that the maximum response of a structure is equal to the sum of the maximum response in each of its mode. Furthermore, as a response spectrum does not provide information about the time at which each maximum occurs, it is not possible to derive an exact formula to compute the maximum response in terms of the modal maxima. One of the first rules for the combination of modal response was the so called **SQUARE ROOT OF THE SUM OF THE SQUARES RULE (SRSS)**.

\[
max\{u(t)\} = \sqrt{\sum_i (\{\Phi\}_i^T \max(\eta_i(t)))^2}
\]
Conclusions

• A structure can be considered as a filter that modifies the input signal (ground motion) into an output signal (displacement, acceleration, velocity).

• Hence the response of the structure depends on the characteristics of the ground motion and on its dynamic behaviour (natural period and damping). A ground motion can be very hard for some structures but not so for some others!!!

• The response spectrum is a very useful tool to evaluate the maximum relative displacement or maximum absolute acceleration.

• A structure can be seen as a N-DOF system, whose response can be calculated as the linear combination of N independent S-DOF systems.