Reliability analysis in engineering applications

Extreme value distributions
Max and Min

Extreme value distributions

The highest value during the reference period

Variable load, Q

Reference period

Time

Extreme value distributions

The reference period is the time interval the extreme value is determined. Generally the reference period is set to 1 year.

Reasons to divide into reference periods:
1. Climate-related loads, e.g. snow and wind are best described by a random process. To divide them into reference periods makes them stationary and much easier to use for our purposes.
2. If the length of the reference period should be a year or longer, loads can be considered as independent R.V.
3. The length of the reference period is strongly related to the safety index and characteristic values for loads.

Extreme value distributions

Assume that the gross weight of a lorry (7 axles) is normal distributed (parent distribution) with mean value and cov of 40 tons and 15 % respectively.

Determine the extreme value distributions if a bridge is trafficked by 20000 and 80000 such lorries during a year?
Extreme value distributions

Return period, “Återkomstperiod”

Characteristic values for loads are often defined as the 98th percentile of the extreme value distribution.

If the reference period is chosen to 1 year, there will be 2% probability that the characteristic load is exceeded every year.

Classical extreme value theory

The classical extreme value theory descends from a distribution, $M_n$, such as

$$M_n = \max\{X_1, X_2, \ldots, X_n\}$$

Where $X_1, X_2, \ldots, X_n$ independent identically distributed R.V. from a distribution $X$. $X$ is called the "parent distribution".

Classical extreme value theory

How is $M_n$ distributed?

$$F_{M_n}(x) = \left[F_X(x)\right]^n$$

where $F_X$ is the parent distribution for the R.V. is $X$.

It can be shown that the equation converts to an asymptotic distribution when $n \to \infty$.

If $n$ increases, then the mean value increases and the variation decreases.
Classical extreme value theory

There is a family of three such distributions:

Typ 1, Gumbel
\[ G(x) = \exp \left( -\exp \left( \frac{x - b}{a} \right) \right) \]
\[ x \leq b \]
\[ G(x) = 0 \]
\[ x > b \]

Typ 2, Fréchet
\[ G(x) = \exp \left( -\left( \frac{x - b}{a} \right)^{-k} \right) \]
\[ x < b \]
\[ G(x) = \exp \left( -\left( \frac{x - b}{a} \right)^{-k} \right) \]
\[ x \geq b \]

Typ 3, Weibull
\[ G(x) = \exp \left( -\left( \frac{x - b}{a} \right)^k \right) \]
\[ x < b \]
\[ G(x) = \left( \frac{x - b}{a} \right)^k \]
\[ x \geq b \]

\[ a, b \text{ and } k \text{ are the scale, location and form parameter respectively.} \]

The general extreme value distribution, ”GEV”

The three extreme value distributions can be combined into one distribution called the “general extreme value distribution”, GEV, such as:

\[ F_x(x,a,b,k) = \begin{cases} \exp(-(1-(x-b)/a)^{1/k}) & \text{if } k \neq 0 \\ \exp(-(x-b)/a) & \text{if } k = 0 \end{cases} \]
\[ a > 0 \]

\[ a, b \text{ and } k \text{ are parameters of the GEV-distribution.} \]

The general extreme value distribution, ”GEV”

The parameter \( k \) is often called "extreme value index".

If \( X_1, X_2, \ldots, X_n \) are independent identical distributed GEV R.V. with parameters as below.

\[ k \neq 0 \]
\[ a \]
\[ b = \frac{a}{k} \ln(n) \]

\[ k = 0 \]
\[ a \]

\[ b \]

\[ F_X(x) \text{ and } f_X(x) \text{ are the cumulative dist. func. and the prob. dist. func. of the normal dist.} \]

Extreme values distributions

Two special cases:

Maximum of a normal or exponential distributed variables both becomes Gumbel distributed with cumulative distribution function as below.

\[ F_g(x) = e^{-(x-a)} \quad \text{for } k \neq 0 \]
\[ b = a \ln(\text{mean}) \]
\[ a \text{ and } b \text{ are parameters of the Gumble dist.} \]

Exponential dist.

\[ F_x(x) = 1 - e^{-x/b} \]
\[ a = b \]
\[ b = \ln(\text{mean}) \]

Normal dist.

\[ F_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp \left( -\frac{(x-\mu)^2}{2\sigma^2} \right) \, dx \]
\[ a = \mu \]
\[ b = \sigma \]
\[ F_x \text{ and } f_x \text{ are the cumulative dist. func. and the prob. dist. func. of the normal dist.} \]
Peaks Over Thresholds, (POT)

A method to estimate quantiles outside the range of data.

Moments generated by vehicles in a simple supported 30 m span bridge.

Peaks Over Thresholds, (POT)

Emperical distribution

Peaks Over Thresholds, (POT)

Fit of \((\text{moments}-u)\) for \((\text{moments}>u)\) to a exponential dist.

Peaks Over Thresholds, (POT)

It can be shown that maximum of a Poisson distributed number of exponential distributed exceedances over the threshold is Gumbel distributed with parameters as below.

\[
\lambda = \frac{\ln(\lambda_u)}{u - \lambda u} \\
\lambda_u: \text{intensity of events that exceeds the threshold}
\]

Peaks Over Thresholds, (POT)

The general Pareto distribution

\[
F_X(x|\xi,\alpha) = 1 - \left(\frac{x-\xi}{\alpha}\right)^{-\alpha}, \\
\xi, \alpha > 0 \\
u \text{ and } \lambda \text{ are the threshold and the expected number of exceedances.}
\]

Extremvärde

Another useful way to determine extreme value distributions is through simulations.